Trigonometric equations

1) Equations form:

$$a\sin^2 x + b\sin x + c = 0$$
$$a\cos^2 x + b\cos x + c = 0$$
$$atg^2 x + btgx + c = 0$$
$$actg^2 x + bctgx + c = 0$$

Resolve with sthe replacement t

Of course, equation $at^2 + bt + c = 0$ has real solutions for $D \geq 0$. When you find t_1, t_2, went back on replacement and pay attention, because with $\sin x$ and $\cos x$ must be $|t_1| \leq 1$ i $|t_2| \leq 1$. While tgx and $ctgx$ can take value from the entire set of real numbers.

Examples:

Solve the equations:

a) $2\sin^2 + 3\sin x + 1 = 0$

b) $2\cos^2 x - 7\cos x + 3 = 0$

c) $tg^2 x - 3tgx + 2 = 0$

d) $2ctgx + tgx = 3$

e) $2\sin^2 x - \cos x = 1$

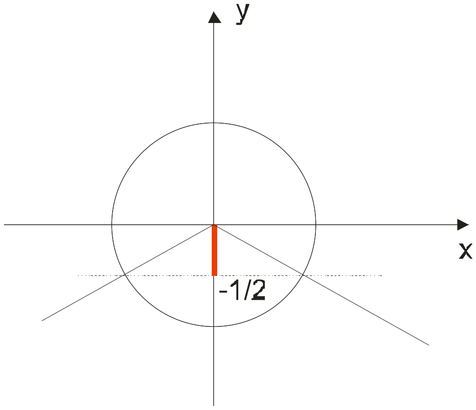
Solution:

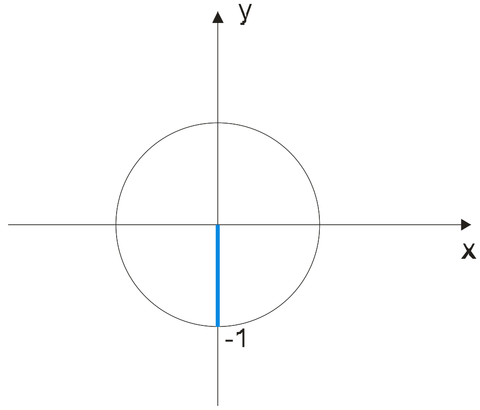
a) $2\sin^2 + 3\sin x + 1 = 0 \Rightarrow$ replacement $\sin x = t$

$$2t^2 + 3t + 1 = 0$$
$$t_{1,2} = \frac{-3 \pm 1}{4}$$
$$t_1 = -\frac{1}{2}$$
$$t_2 = -1$$

Back in the replacement:

$\sin x = -\frac{1}{2}$ or $\sin x = -1$



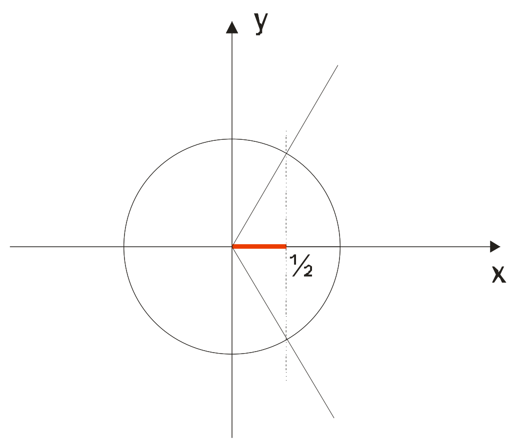


$$x_1 = -\frac{\pi}{6} + 2k\pi$$
$$x_2 = \frac{7\pi}{6} + 2k\pi$$
$$k \in Z$$

$$x_3 = -\frac{\pi}{2} + 2k\pi$$
$$k \in Z$$

b)

$$2\cos^2 x - 7\cos x + 3 = 0 \to replacement ... \cos x = t$$
$$2t^2 - 7t + 3 = 0$$
$$t_{1,2} = \frac{7 \pm 5}{4}$$
$$t_1 = 3 \to impossible, because -1 \le \cos x \le 1$$
$$t_2 = \frac{1}{2}$$



So, solutions are:

$$x_1 = \frac{\pi}{3} + 2k\pi$$
$$x_2 = -\frac{\pi}{3} + 2k\pi$$
$$k \in Z$$

c) $tg^2x - 3tgx + 2 = 0 \longrightarrow tgx = t$

$t^2 - 3t + 2 = 0$

$t_{1,2} = \frac{3 \pm 1}{2}$

$t_1 = 2$

$t_2 = 1$

$tgx = 2$ or $tgx = 1$

$x_1 = arctg2 + k\pi$ $x_2 = \frac{\pi}{4} + k\pi, k \in Z$

d) 2ctgx+tgx+3 $\rightarrow$ $tgx = \frac{1}{ctgx}$

2ctgx+$\frac{1}{ctgx} = 3 \rightarrow$ replacement : $ctgx = t$

$2t + \frac{1}{t} = 3$

$2t^2 - 3t + 1 = 0$

$t_{1,2} = \frac{3 \pm 1}{4}$

$t_1 = 1$

$t_2 = \frac{1}{2}$

For $ctgx = 1$ is $x_1 = \frac{\pi}{2} + k\pi, k \in Z$

For $ctgx = \frac{1}{2}$ is $x_2 = arcctg\frac{1}{2} + k\pi$

e) $2\sin^2 x - \cos x = 1$

Here we must all "move" in sinx or cosx. It is easier to use $\sin^2 x = 1 - \cos^2 x$ and move to cosx.

$2(1 - \cos^2 x) - \cos x - 1 = 0$

$2 - 2\cos^2 x - \cos x - 1 = 0$

$-2\cos^2 x - \cos x + 1 = 0 / \cdot (-1)$

$2\cos^2 x + \cos x - 1 = 0 \rightarrow replacement : \cos x = t$

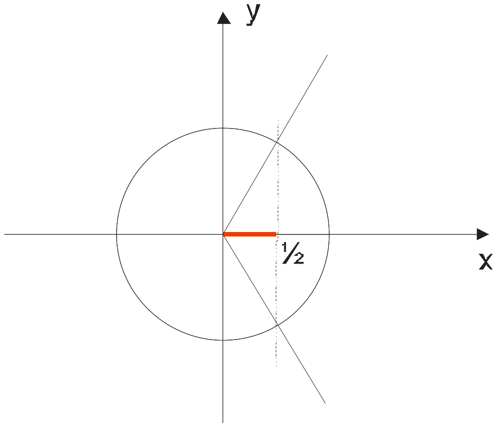
$2t^2 + t - 1 = 0$

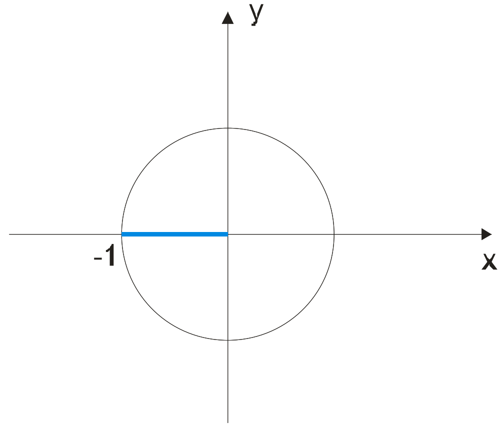
$t_{1,2} = \frac{-1 \pm 3}{4}$

$t_1 = \frac{1}{2}$

$t_2 = -1$

$\cos x = \frac{1}{2}$ or $\cos x = -1$





$x_1 = \frac{\pi}{3} + 2k\pi$

$x_2 - \frac{\pi}{3} + 2k\pi$

$k \in Z$

or

$x_3 = \pi + 2k\pi$

$k \in Z$

2) **Homogeneous equation**

It is form: $\boxed{a\sin^2 x + b\sin x\cos x + c\cdot\cos^2 x = 0}$

We solved this equation with replacement. First devide all equation with $\cos^2 x$ and then we have form $atg^2x + btgx + c = 0$. Pay attention, because $\sin x \neq 0$ and $\cos x \neq 0$.

In a similar way , we solve equation $\boxed{a\sin^2 x + b\sin x\cos x + c\cdot\cos^2 x = d}$

Write as a "trick" that: $d = d\cdot 1 = d\cdot(\sin^2 x + \cos^2 x)$, all switch to the left side and we have:

$(a-d)\sin^2 x + b\sin x\cos x + (c-d)\cos^2 x = 0$ Solve this equation as homogenous…

Examples:

Solve the equations:

a) $2\sin^2 x - 5\sin x\cos x + 3\cos^2 x = 0$
b) $5\sin^2 x + 2\sin x\cos x + \cos^2 x = 2$

Solution:

a)

$$2\sin^2 x - 5\sin x\cos x + 3\cos^2 x = 0 / : \cos^2 x \neq 0$$

$$2\frac{\sin^2 x}{\cos^2 x} - 5\frac{sonx\cos x}{\cos^2 x} + 3\frac{\cos^2 x}{\cos^2 x} = 0$$

$$2tg^2 x - 5tgx + 3 = 0 \rightarrow replacement(tgx = t)$$

$$2t^2 - 5t + 3 = 0$$

$$t_{1,2} = \frac{5 \pm 1}{4}$$

$$t_1 = \frac{3}{2}$$

$$t_2 = 1$$

For $tgx = \frac{3}{2} \Rightarrow x_1 = arctg\frac{3}{2} + k\pi, k \in Z$

For $tgx = 1 \Rightarrow x_2 = \frac{\pi}{4} + k\pi, k \in Z$

b)

$$5\sin^2 x + 2\sin x\cos x + \cos^2 x = 2$$

$$5\sin^2 x + 2\sin x\cos x + \cos^2 x = 2(\sin^2 x + \cos^2 x)$$

$$5\sin^2 x + 2\sin x\cos x + \cos^2 x = 2\sin^2 x + 2\cos^2 x$$

All switch to the left side!

$$3\sin^2 x + 2\sin x\cos x - \cos^2 x = 0 / : \cos^2 x$$

$$3tg^2 x + 2tgx - 1 = 0 \rightarrow (tgx = t)$$

$$3t^2 + 2t - 1 = 0$$

$$t_{1,2} = \frac{-2 \pm 4}{6}$$

$$t_1 = \frac{1}{3}$$

$$t_2 = -1$$

For $tgx = \frac{1}{3} \Rightarrow x = arctg\frac{1}{3} + k\pi, k \in Z$

For $tgx = -1 \Rightarrow x = -\frac{\pi}{4} + k\pi, k \in Z$

3) <u>Equations form:</u>

$$\begin{matrix}\sin ax \pm \sin bx = 0 \\ \cos ax \pm \cos bx = 0\end{matrix} \quad \text{and alike….}$$

In this equations we first use the formulas:

1. $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
2. $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
3. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
4. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

After that: $A \cdot B = 0 \Leftrightarrow A = 0 \vee B = 0$

Examples:

Solve the equations:

a) $\sin 6x - \sin 4x = 0$
b) $\cos 3x + \cos x = 0$
c) $\sin x = \cos 2x$
d) $\sin x + \sin 2x + \sin 3x = 0$

Solution:

a)

$$\sin 6x - \sin 4x = 0$$

$$2\cos\frac{6x+4x}{2}\sin\frac{6x-4x}{2} = 0$$

$$2\cos 5x \cdot \sin x = 0$$

$$\cos 5x = 0 \vee \sin x = 0$$

$$5x = \frac{\pi}{2} + 2k\pi \qquad 5x = -\frac{\pi}{2} + 2k\pi$$

$$x = \frac{\pi}{10} + \frac{2k\pi}{5} \qquad x = -\frac{\pi}{10} + \frac{2k\pi}{5}$$

$$x = \frac{\pi}{10} + \frac{k\pi}{5}$$

$$k \in Z$$

$$\sin x = 0$$

$$x = 0 + 2k\pi \qquad x = \pi + 2k\pi$$

$$x = k\pi$$

$$k \in Z$$

b)

$$\cos 3x + \cos x = 0$$

$$2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2} = 0$$

$$2\cos 2x \cos x = 0$$

$\cos 2x = 0$ or $\cos x = 0$

$$2x = \frac{\pi}{2} + 2k\pi \qquad 2x = -\frac{\pi}{2} + 2k\pi$$

$$x = \frac{\pi}{4} + k\pi \qquad x = -\frac{\pi}{4} + k\pi$$

$$x = \frac{\pi}{4} + \frac{k\pi}{2}$$

$$k \in Z$$

$$x = \frac{\pi}{2} + 2k\pi \qquad x = -\frac{\pi}{2} + 2k\pi$$

$$x\frac{\pi}{2} + k\pi$$

$$k \in Z$$

c)

$$\sin x = \cos 2x$$

$$\sin x - \cos 2x = 0$$

$$\sin x - \sin(\frac{\pi}{2} - 2x) = 0$$

$$2\cos\frac{x + \frac{\pi}{2} - 2x}{2}\sin\frac{x - (\frac{\pi}{2} - 2x)}{2} = 0$$

$$2\cos(\frac{x}{2} - \frac{\pi}{4})\sin(\frac{3x}{2} - \frac{\pi}{4}) = 0$$

From here we have:

$\cos(\frac{x}{2} - \frac{\pi}{4}) = 0$ or $\sin(\frac{3x}{2} - \frac{\pi}{4}) = 0$

For $\cos(\frac{x}{2} - \frac{\pi}{4}) = 0$ is:

$\frac{x}{2} - \frac{\pi}{4} = \frac{\pi}{2} + 2k\pi$ or $\frac{x}{2} - \frac{\pi}{4} = -\frac{\pi}{2} + 2k\pi$

$$\frac{x}{2} = \frac{\pi}{4} + \frac{\pi}{2} + 2k\pi \qquad \frac{x}{2} = -\frac{\pi}{4} + \frac{\pi}{2} + 2k\pi$$

$$\frac{x}{2} = \frac{3\pi}{4} + 2k\pi \qquad \frac{x}{2} = -\frac{\pi}{4} + 2k\pi$$

$$x = \frac{3\pi}{2} + 4k\pi \qquad x = -\frac{\pi}{2} + 4k\pi$$

Together: $x = \frac{3\pi}{2} + 2k\pi$

For $\sin(\frac{3x}{2}-\frac{\pi}{4})=0$ is:

$$\frac{3x}{2}-\frac{\pi}{4}=0+2k\pi$$

$$3x-\frac{\pi}{2}=4k\pi$$

$$3x=\frac{\pi}{2}+4k\pi$$

$$x=\frac{\pi}{6}+\frac{4k\pi}{3}$$

or

$$\frac{3x}{2}-\frac{\pi}{4}=\pi+2k\pi$$

$$3x-\frac{\pi}{2}=2\pi+4k\pi$$

$$3x=2\pi+4k\pi$$

$$3x=2\pi+\frac{\pi}{2}+4k\pi$$

$$x=\frac{5\pi}{6}+\frac{4k\pi}{3}$$

Together : $x=\frac{\pi}{6}+\frac{2k\pi}{3}$

d)

$$\sin x+\sin 2x+\sin 3x=0$$

$$2\sin\frac{x+3x}{2}\cos\frac{x-3x}{2}+\sin 2x=0$$

$$2\sin 2x\cos x+\sin 2x=0$$

$$\sin 2x(2\cos x+1)=0$$

$\sin 2x=0$ or $2\cos x+1=0$

$$2x=0+2k\pi \qquad x=k\pi$$

$$2x=\pi+2k\pi \qquad x=\frac{\pi}{2}+k\pi$$

$$x=\frac{2\pi}{3}+2k\pi \qquad x=\frac{4\pi}{3}+2k\pi$$

4) Equation form: $a\sin x+b\cos x=c$

This equation can be solved in several ways:

i) replacement $tg\frac{x}{2}=t$

ii) by introducing support argument

iii) the method of creating a system

i) **replacement** $tg\frac{x}{2}=t$

$\sin x=\frac{2t}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2} \rightarrow$ already done earlier!

After replacament, we get the square equation by t.

Problem may occur when we are looking for zero of received polynomials. This method we will do when we do not have other options. Of course, since the replacement is $tg\frac{x}{2}=t$, must be $x \neq \pi+2k\pi$

ii) Introducing support argument

$a \cdot \sin x + b \cdot \cos x = c$

Introduce a new argument φ by:

$$\sin\varphi=\frac{b}{\sqrt{a^2+b^2}}, \cos\varphi=\frac{a}{\sqrt{a^2+b^2}} \Rightarrow \varphi = arctg\frac{b}{a}, \rightarrow tg\varphi=\frac{b}{a}$$

And starting equation down to:

$$\sin(x+\varphi)=\frac{c}{\sqrt{a^2+b^2}}$$

Of course, must be: $\left|\frac{c}{\sqrt{a^2+b^2}}\right| \leq 1 \longrightarrow a^2+b^2 \geq c^2$.

If $a^2+b^2 < c^2$, equation have no solution!

iii) the method of creating a system

We have equation $a\sin x + b\cos x = c$, and add here $\sin^2 x + \cos^2 x = 1$

From the first equation express sinx or cosx and change to another.

Examples:

Solve the equations:

a) $\sin x + \sqrt{3}\cos x = 2$
b) $2\sin x + 5\cos x = 4$

Solution:

a) First, we try the method of support argument:

$$\sin x + \sqrt{3}\cos x = 2 \Rightarrow a = 1, b = \sqrt{3}, c = 2$$

$$a^2 + b^2 = 1^2 + \sqrt{3}^2 = 1 + 3 = 4$$ condition : $a^2 + b^2 \geq c^2$, $4 \geq 4$ met!

$$c^2 = 4$$

$$\varphi = arctg\frac{b}{a}$$

$$\varphi = arctg\frac{\sqrt{3}}{1}$$

$$tg\varphi = \sqrt{3} \Rightarrow \varphi = 60^o = \frac{\pi}{3}$$

$$\frac{c}{\sqrt{a^2 + b^2}} = \frac{2}{\sqrt{1^2 + \sqrt{3}^2}} = \frac{2}{2} = 1$$

$$\sin(x + \varphi) = \frac{c}{\sqrt{a^2 + b^2}}$$

$$\sin(x + \frac{\pi}{3}) = 1$$

$$x + \frac{\pi}{3} = \frac{\pi}{2} + 2k\pi$$

$$x = \frac{\pi}{2} - \frac{\pi}{3} + 2k\pi$$

$$x = \frac{\pi}{6} + 2k\pi$$

$k \in Z$

So, this method is "good". Always try it first!

b)

$$2\sin x + 5\cos x = 4$$
$$a = 2, b = 5, c = 4$$

condition : $a^2 + b^2 \geq c^2$, $4 + 25 \geq 16$, met!

$$tg\varphi = \frac{b}{a}$$
$$tg\varphi = \frac{5}{2}$$

Here's the problem! We can not easily find the angle!

Try the third method, with the system:

$$2\sin x + 5\cos x = 4 \Rightarrow \cos x = \frac{4 - 2\sin x}{5}$$
$$\sin^2 x + \cos^2 x = 1$$
$$\sin^2 x + \left(\frac{4 - 2\sin x}{5}\right)^2 = 1$$
$$\sin^2 x + \frac{16 - 16\sin x + 4\sin^2 x}{25} = 1$$
$$25\sin^2 x + 16 - 16\sin x + 4\sin^2 x = 25$$
$$29\sin^2 x - 16\sin x - 9 = 0 \to replacement(\sin x = t)$$
$$29t^2 - 16t - 9 = 0$$
$$t_{1,2} = \frac{16 \pm \sqrt{1300}}{58} = \frac{16 \pm 10\sqrt{13}}{58}$$
$$t_1 = \frac{16 + 10\sqrt{13}}{58} = \frac{2(8 + 5\sqrt{13})}{58} = \frac{8 + 5\sqrt{13}}{29} \approx 0{,}896$$
$$t_2 = \frac{16 - 10\sqrt{13}}{58} = \frac{8 - 5\sqrt{13}}{29} \approx -0{,}346$$

So:

$$x = \arcsin\left(\frac{8 + 5\sqrt{13}}{29}\right) + 2k\pi$$
$$x = \arcsin\left(\frac{8 - 5\sqrt{13}}{29}\right) + 2k\pi$$

5) Replacement $\underline{\cos 2x = t}$

If in equation occur terms $\sin^2 x, \cos^2 x, \cos 2x,$ apply this replacement!

Because: $\sin^2 x = \frac{1-t}{2}; \cos^2 x = \frac{1+t}{2}$

Example:

Solve the equation: $8\cos^6 x - 4\sin^4 x = \cos 2x$

Solution:

$$\cos 2x = t$$
$$\cos^6 x = (\cos^2 x)^3 = (\frac{1+t}{2})^3$$
$$\sin^4 x = (\sin^2 x)^2 = (\frac{1-t}{2})^2$$
$$8\cos^6 x - 4\sin^4 x = \cos 2x$$
$$8(\frac{1+t}{2})^3 - 4(\frac{1-t}{2})^2 = t$$
$$8\frac{1+3t+3t^2+t^3}{8} - 4\cdot\frac{1-2t+t^2}{4} = t$$
$$1+3t+3t^2+t^3-1+2t-t^2-t=0$$
$$t^3+2t^2+4t=0$$
$$t(t^2+2t+4)=0 \Leftrightarrow t=0 \vee t^2+2t+4=0$$
$$\cos 2x = 0$$

Solution is:

$$x = \frac{\pi}{4} + k\pi, k \in Z$$

Here are some of the methods for solving trigonometric equation. We need to say that there is no general method for each trigonometric equation.

Try to transform the terms, using the known formula, "creating" a product that will be equal to zero, introduces the appropriate replacement. **Good luck!**